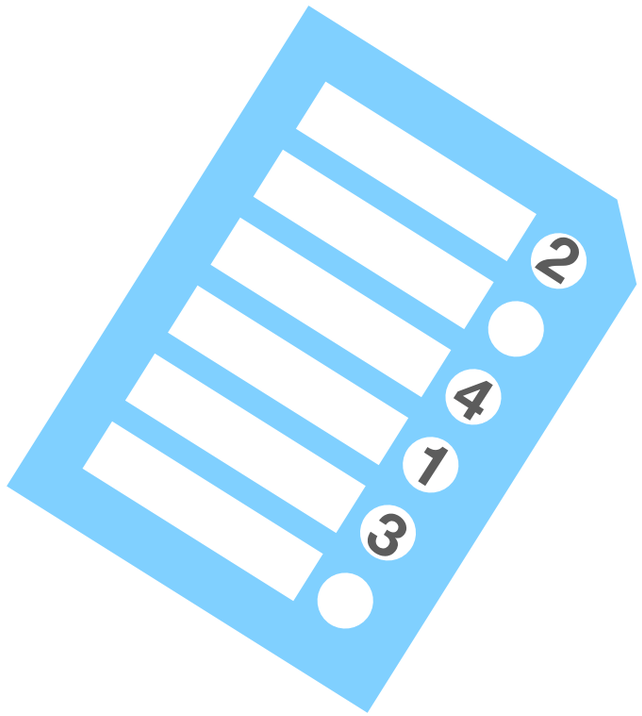
A Formalization of an Instant Run-Off Voting Scheme



Shilpi Goel & Mayank Manjrekar

ACL2-2018 *15 min. Rump Session Talk*

Instant Run-Off Voting

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a.k.a. Single-Winner Ranked Choice Voting

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IRV is a *preferential voting* scheme: voters rank candidates in order of preference to elect one winner.

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Voter 1	A	B	C
Voter 2	A	C	B
Voter 3	C	B	A
Voter 4	B	C	

Where is IRV Used?

- **Politics**
  - US Senate and Congress Race in Maine
  - President of India
  - Mayor of London
  - Members of the Australian Parliament's lower House
- **Entertainment**
  - Oscar's Best Picture Award
- **Computer Science**
  - Planning
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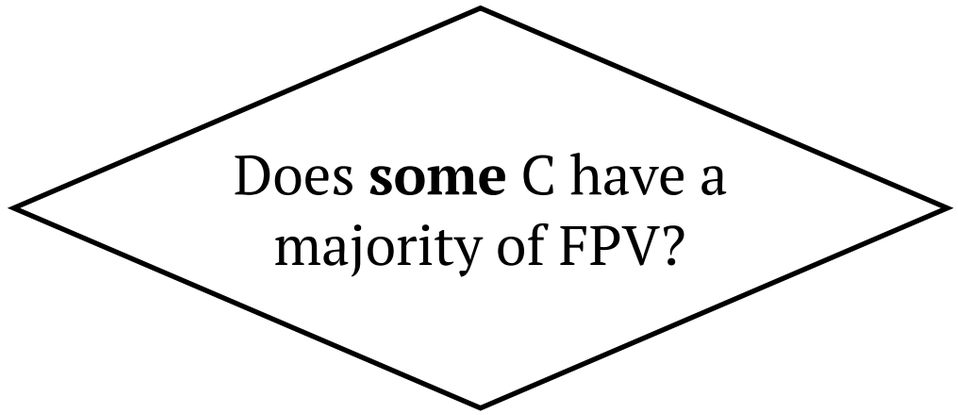
We got interested because of **ACL2-2018's slogan election.**

ACL2-2018: IRV Scheme

C	Candidate
FPV	First-Place Votes

ACL2-2018: IRV Scheme

Majority Step



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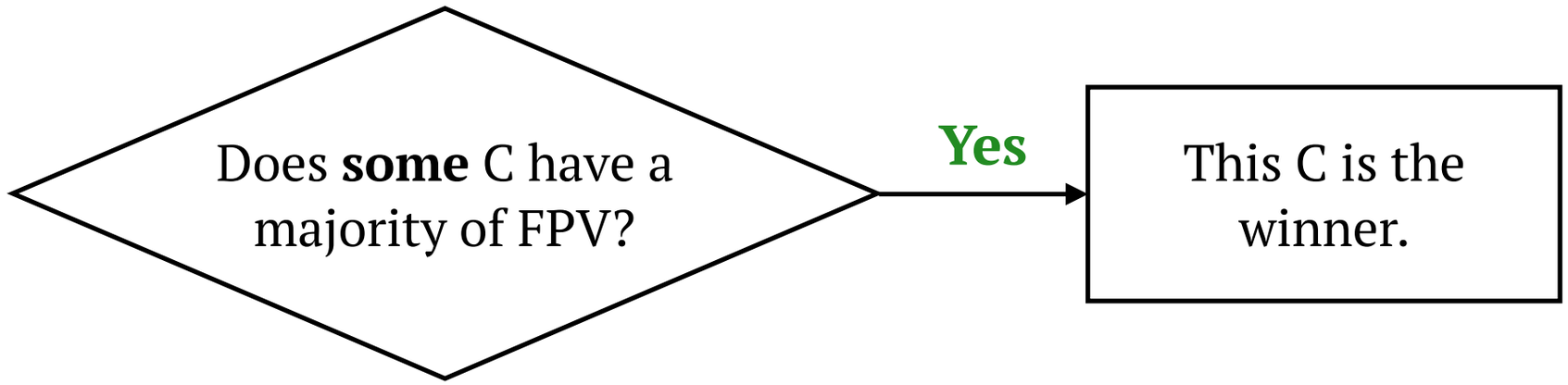

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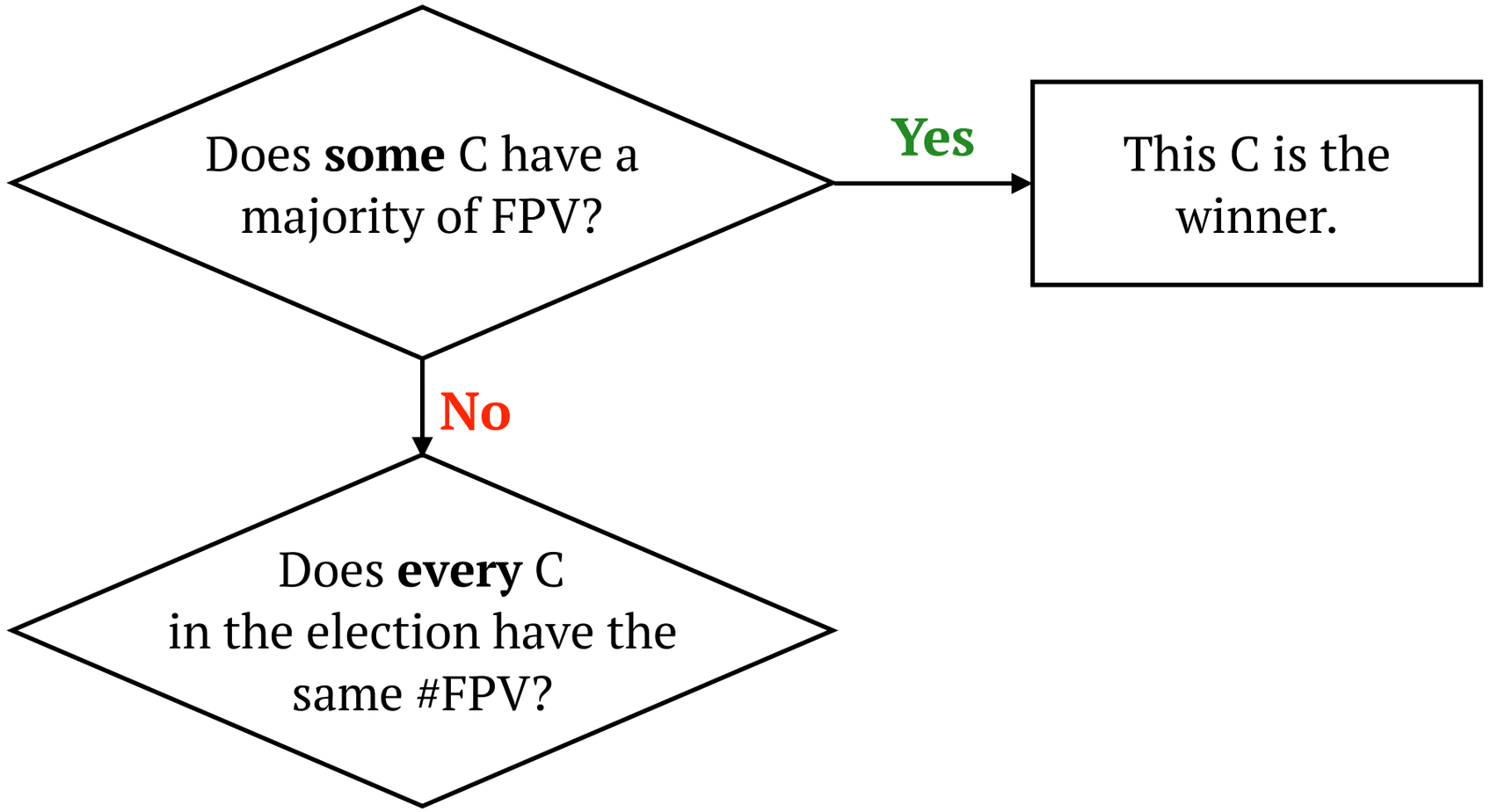

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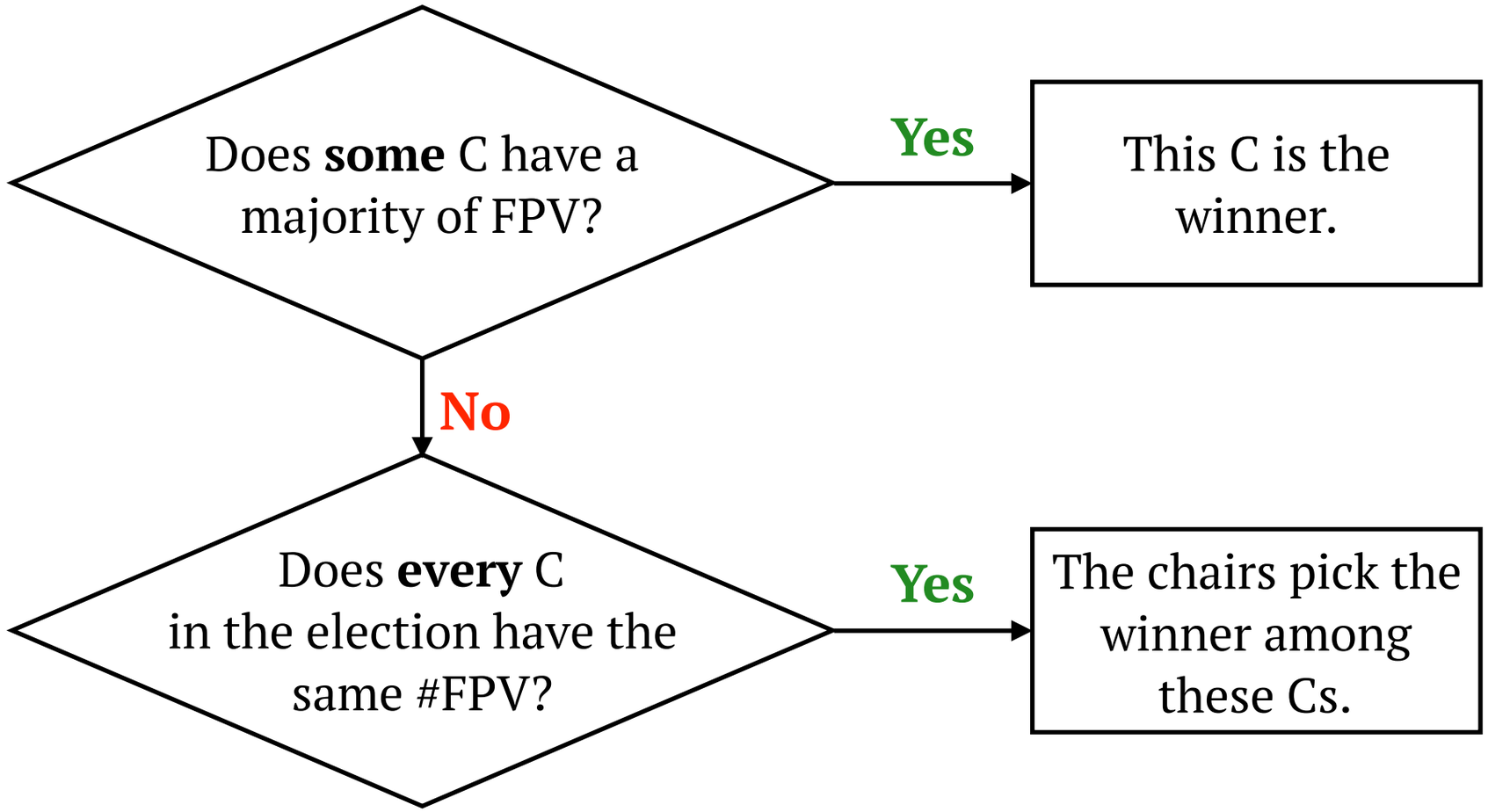

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Does **some** C have a majority of FPV?

Yes → This C is the winner.

No →

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Does **every** C in the election have the same #FPV?

Yes → The chairs pick the winner among these Cs.

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Elimination Step

Eliminate **all** Cs with the least #FPV; **adjust each ballot**.

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"if I were to do this again … if there's a tie for least first-place votes, then it's broken by which of those has the least second-place votes, etc., before deleting candidates."

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[books]/projects/irv

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Pick-Candidate(B,C) = B

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C wins

[books]/projects/irv

[books]/projects/irv

```
(defun irv (xs)
  (cond
   ((or (not (irv-ballot-p xs))
        (endp xs))
    nil)
   (t
    (b* ((cids (candidate-ids xs))
         (maj-winner? (first-choice-of-majority-p cids xs))
         ((when (natp maj-winner?)) maj-winner?)
         (weak-candidate
          (candidate-with-least-nth-place-votes 0 cids xs))
         (new-xs (eliminate-candidate weak-candidate xs)))
      (irv new-xs)))))
```

[books]/projects/irv

```
(defun candidate-with-least-nth-place-votes (n cids xs)
 (cond ((endp cids) nil)

       ((< n (number-of-candidates xs))
        (let* ((relevant-candidates
                (candidates-with-min-votes n cids xs)))
          (if (equal (len relevant-candidates) 1)
              (car relevant-candidates)
            (candidate-with-least-nth-place-votes
              (1+ n) relevant-candidates xs))))

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        ;; Tie persisted throughout all the preference
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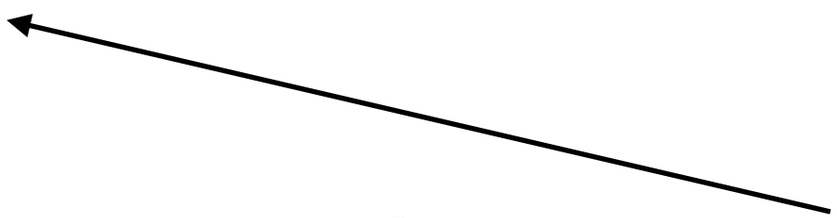

Constrained function:
returns a member of its input

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  - Very rich
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- **Computational Choice Theory**
  - Additional sets of concerns: parallelizability, importance of tie-breaking, etc.

Fairness Criteria

Our formalization meets the following criteria that should be satisfied by IRV schemes:

- **Majority Winner Criterion**
- **Condorcet Loser Criterion**
- **Majority Loser Criterion**

Majority Winner Criterion

If a candidate is preferred by an absolute majority of voters, then that candidate must win.

```
(defthm irv-satisfies-the-majority-criterion
  (implies (and (< (majority (number-of-voters xs))
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Straightforward; needed lemmas like:

- If `e` gets the majority of first-place votes, then there cannot be a tie for the maximum number of first-place votes.

Condorcet Loser Criterion

If a candidate L loses a head-to-head competition against every other candidate, then L must not win the overall election.

```
(defthm irv-satisfies-the-condorcet-loser-criterion
  (implies (and (all-head-to-head-competition-loser-p l cids xs)
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- Otherwise, induct on `xs`.

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  - Note that `w ≠ id`.
  - If `l == id`, then `w ≠ l`.
  - Otherwise, `l` is still the head-to-head loser after `id` is eliminated. So by the induction hypothesis, `w ≠ l`.

Majority Loser Criterion

If a majority of voters prefers every other candidate over a given candidate l, then l must not win.

Note that l has a majority of last-place votes.

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But, a candidate who gets the majority of last-place votes must be the Condorcet Loser.

Conclusion

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:doc irv::instant-runoff-voting
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- Applications:
  - Slogan winner for the next ACL2 Workshop?

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:doc irv::instant-runoff-voting
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- Possible future work: include other properties & schemes
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Thanks!